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Superosculated Sections of Surfaces

Dissertation
Submitted to the Board of University Studies of the Johns Hopkins University for the degree of Doctor of Philosophy by

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The object of the following pages is to study sections of a surface which are superosculated by circles of [illegible] Lines of curvature [illegible] are under consideration, but, in addition, a short study of osculating lines is made, with special reference to conditions [illegible] on certain classes of surfaces.

These lines were first treated by De la Gournerie in an article published in

Liouville Journal de Mathématiques 1re Série Tome XX 1855

In 1875, Ribaucour published a note in Vol 80 of the Comptes Rendus giving a [illegible] of the [illegible] of [illegible]

The subject was undertaken at the suggestion of Prof. [illegible], whose kindly interest in the work has helped me very much at critical times.

above will be used as parameters of lines. The linear element will take the form

$$ds^2 = E\,du^2 + G\,dv^2$$

coordinates are used, the origin will be taken at a point on the surface, the tangents to the u and v lines will be the X and Y axes respectively and the normal to the surface will be the Z axis.

subject I will draw attention to a few formulae which will be used from time to time in the course of the work

They are [illegible]
by Darboux in Book V of his
"Théorie Générale des Surfaces."
His notation will be retained.

Having given a curve C,
through any point M of
a surface, let us denote by
ρ, ρ_g and τ the radii of
curvature, of geodesic curv-
ature and of torsion, respect-
ively, of C; by ω, the angle
between the X axis and
the tangent to the curve, and
by ϖ, the angle between
the X axis and the principal
normal to the curve. Then,
[illegible]
radii of curvature of the
surface at the point M, in

Show the relations,

$$\frac{\sin\bar{\omega}}{\rho} = \frac{1}{\rho_g} = \frac{d\omega}{ds} - \frac{\cos\omega}{\sqrt{EG}}\frac{\partial E}{\partial v} + \frac{\sin\omega}{\sqrt{EG}}\frac{\partial G}{\partial u}.$$

The centre of the osculating sphere of the curve C is determined by the intersection of three consecutive normal

planes. These three planes have the same point in common when the three following conditions have, namely,

$$\bar{X}\cos\omega + \bar{Y}\sin\omega = 0$$

$$\left(\frac{\bar{X}\cos\omega}{\bar{R}} + \frac{\bar{Y}\sin\omega}{\bar{R}'}\right) + \frac{d}{dz}\left(\bar{X}\sin\omega - \bar{Y}\cos\omega\right)$$

The determinant Δ of these three equations becomes, after reducing equations (1) to (1)

$$\Delta = \frac{1}{\rho^2 r}.$$

Hence if we consider a plane section through the point M, $\frac{1}{r} = 0$ and

the reciprocal radius
would in general the centre
of the sphere goes to infinity
that is, the sphere becomes
merely the plane of section.
If however we consider only
superosculated sections, that
is, sections which have contact
osculating circles the three
planes (S) must have a line
in common, any point of
which is the centre of an
osculating sphere of (S)

We have then only two eq
to find the centre; that is
X Y and Z become indeterminate

so that we have

$$\frac{dp}{ds} + \frac{3}{\rho g}\left(\frac{1}{R} - \frac{1}{R'}\right)\sin\omega\cos\omega = 0$$

$$\frac{d\frac{1}{\rho g}}{ds} + \frac{\cos\omega}{\rho}\left(\frac{1}{R} - \frac{1}{R'}\right)\sin\omega\cos\omega = 0$$

Equations (6) are found to be the equivalent to the single condition

[illegible]

Proved

For, if (4) holds we find that equation (4) reduces [illegible] and

determinate. This involves that the section be superosculated

by a circle, and sufficient that a plane section be superosculated at a point, in that d^2 vanish at the point in question. We have the relations

$$\cos\omega = \frac{dx}{ds}$$

$$\sin\omega = \frac{dy}{ds} = \sqrt{G}$$

In case a section is superosculated, (1) takes the form

$$\frac{1}{R'} = \frac{\partial R}{\partial u}\cos^3\omega$$

only one value of $\bar{w}$; that is, any curve through M

On the other hand, for any definite value of $\bar{w}$, w will take one of eight values; that is, there are

that is take normal sections there of the eight roots are double roots, the remaining two arise from the factor

$$\frac{1}{(Hdu)^2} = 0$$

i.e. $x^2 + y^2 = 0$.

So that at any point, the circular rays be infinitely determine normal sections which are superosculated. Leaving these out of account, we have for it

$$\frac{\rho}{q} = 0$$

and $\therefore \quad \nabla = 0$. that is

$$\frac{E}{R^2}\frac{\partial R}{\partial u}du^3 + \frac{3E'}{R^2}\frac{\partial R}{\partial u}dudv + \frac{3B}{R^2}\frac{\partial R'}{\partial u}dudv + \frac{B}{R^2}\frac{\partial R'}{\partial v}dv^3 -$$

A superosculating [illegible]
curve, at each of whose
points, the normal vector
tangent to it, is superosculated by a circle.

superosculating [illegible]

which must be real.

This theorem might have

from the equation -

$$\rho_n = \frac{D du^2 + D'' dv^2}{ds^2}$$

Differentiate this and
[illegible] the result to zero
and you get precisely the
equation (11)

Darboux's relation

lines are retained as we pass from any surface to a parallel one.

We have seen that one of the families of super-osculating lines is always real. What is the condition that all these should be real?

An obvious class of surfaces on which we have this is canal surfaces, for which one of the principal

radius is constant. Say

$$R = \text{const.}$$

In this case equation (10) reduces to

$$\left[3\frac{\partial R'}{\partial u}du + \frac{\partial R'}{\partial v}dv\right]dv^2 = 0$$

and the osculating lines are the u lines taken twice together with the family

$$3\frac{\partial R'}{\partial u}du + \frac{\partial R'}{\partial v}dv = 0.$$

On the another ring in addition to $R' = 0$, we have $R' = U$ where U is a function of u alone. Hence on an anchor ring these families coincide with

third family a
allels.

Another special case of
canal surface is the developable, on which we have

$$K' = 0$$

So that on a developable lines of the families coincide with the rectilinear generator

The necessary and sufficient condition that equation
that the discriminant be

$$\frac{EG}{R^2}\left(\frac{\partial R}{\partial u}\right)^2\left(\frac{\partial R'}{\partial v}\right)^2 - 6\frac{FG}{R'R''}\frac{\partial R}{\partial u}\frac{\partial R}{\partial v}\frac{\partial R'}{\partial u}\frac{\partial R'}{\partial v} + \frac{GE}{R''^2}\left(\frac{\partial R'}{\partial u}\right)^2$$

$$\frac{E}{R^2}\left(\frac{\partial R}{\partial u}\right)^2 - \frac{2F}{R'R''}\frac{\partial R}{\partial u}\frac{\partial R'}{\partial v} + \frac{G}{R''^2}\left(\frac{\partial R'}{\partial v}\right)^2 = 0$$

This condition will be true if we have

$$\frac{E}{R^2}\frac{\partial R}{\partial u} = \frac{G}{R'^2}\frac{\partial R'}{\partial v} = 0.$$

Leaving out of consideration the case of developables, which has already been discussed, this involves

$$R = V$$
$$R' = U$$

where U and V are functions of u and v respectively. (12) holds when the lines of curvature of the third family belong to the equation

$$E\,dR^2 + G\,dR'^2 = 0.$$

For surfaces of revolution
the u and v lines being
meridians and parallels
we have

$$\frac{\partial U}{\partial v}$$

Hence the parallels form
one system of curves satisfying
lines the other two being

$$\frac{E}{R^2}du^2 + \frac{2F}{R^2}dudv + \frac{G}{R^2}\frac{dR}{du}dv^2 = 0$$

The roots of this are
necessarily real the condition

$$4\frac{G}{R^2}\frac{dR}{du}\frac{dR'}{du} - 3\frac{1}{R^2}\left(\frac{dR}{du}\right)^2 < 0$$

Obviously this will be
true if u is v.

$$\frac{\partial R}{\partial u}\frac{\partial R'}{\partial u} = 0$$

This condition is sufficient for surfaces of revolution but it is not necessary. It will be found to be both necessary and sufficient for a large class of surfaces, namely, Liouville surfaces.

For these we have —

$$\frac{\partial(R R')}{\partial(u v)} = 0$$

Solving this, to eliminate $\frac{\partial R'}{\partial u}$ from the discriminant (11) we get as the condition for three real families of superosculating lines

$$\frac{\partial R}{\partial u}\frac{\partial R'}{\partial u}\left|\frac{E}{R^2}\left(\frac{\partial R}{\partial v}\right)^2 - \frac{G}{R^2}\frac{\partial R}{\partial u}\frac{\partial R'}{\partial u}\right|^2 < 0$$

So that on all Minimgarten surfaces for which as have

$$\frac{\partial R}{\partial u}\,\frac{\partial R'}{\partial u} < 0$$

real.

which $\frac{\partial R}{\partial v}\,\frac{\partial R'}{\partial v}$

If we had taken the symmetric condition for Minimgarten surfaces.

we would have found as a condition for reality

$$\frac{dR'}{dP} < 0.$$

This gives us a criterion which is readily applied

in any special case.

For example, in a surface of constant curvature

$$\frac{1}{R R'} = k$$

$$\frac{dR'}{dR} = -\frac{R'}{R}$$

Therefore, for all surfaces of constant positive curvature all three of the osculating lines are real; for surfaces of constant negative curvature two of the families are imaginary.

For a surface of constant

$$\frac{1}{R} + \frac{1}{R'} = k$$

$$\frac{dR'}{dR} = -\left(\frac{R'}{R}\right)^2$$

Here are three real [illegible] of
superconvexity [illegible]
This includes minimal sur-
faces as a special case for
which we have $K = 0$.

As a last example, let us
look at the surface [illegible]

$$R' = R^* - \frac{2}{K} \tan \frac{K(R+R')}{2}$$

$$\frac{dR'}{dR} = \frac{2 \tan^2 \frac{K(R+R')}{2}}{1 + \cos K(R+R')} = \tan^2 \frac{K(R+R')}{2}$$

[illegible] positive quantity, we know
that [illegible]
[illegible]

$$\frac{E}{\partial v}\left[\frac{G}{\partial v}\frac{\partial R}{\partial u}\frac{\partial R'}{\partial u} - \frac{E}{\partial v}\left|\frac{\partial R}{\partial v}\right|^2\right] du^2 + \frac{EG}{\partial v}\frac{\partial (RR')}{\partial u} du\,dv$$

Thus has the same discriminant as the original cubic and the resultant of the two forms is the square of this common discriminant. Hence, if on any surface

determined by (13) are

If, however, two of the roots of the cubic are identical then the has this same double root.

the Riccatian reduces to -

$$\frac{dR'}{du} + \frac{R'^2}{R^2} = 0$$

In case the surface is one of constant mean curvature this form breaks up simply into lines of length zero namely

$$E\,du^2 + G\,dv^2 = 0$$

Let us examine a few surfaces and calculate their superosculating lines. Take first the catenoid, a minima surface of revolution.

$$ds^2 = du^2 + (u^2 + a^2)\,dv^2$$

$$R' \quad R'' = -\frac{u^2 + a^2}{a}$$

in equation (10) we have

$$du\,[du^2 - (u^2+a^2)\,dv^2] = 0$$

i.e. $du\,[\Delta du^2 + \Delta'' dv^2] = 0$

On the catenoid the asymptotic lines bisect the lines of curvature. Hence we say, on the catenoid the osculating lines are the parallels together with the directors of the lines of curvature.

[illegible] with plane director

Referred to its asymptotic lines, we have

$$ds^2 = du^2 + (u^2+a^2)\,dv^2.$$

The lines of curvature are

$$\Delta' dv^2 + 2\Delta du^2 = 0$$

So that on applying the two surfaces those lines of curvature correspond to the superosculating lines of the conoid

On transforming to the lines of curvature, we have

$$E = G. \qquad D = -D''$$

$$R = -R' = a_{11}\left(\frac{e^{u+v}}{+} \cdot e^{u-v}\right)^2$$

Carrying these values into

$$(du\ dv)(du^2 + 4du\,dv + dv^2) = 0.$$

$$\frac{dv}{du} = \frac{dy}{dx}.$$

Therefore the superosculating lines cut the lines of curvature at constant angles.

On examining (11) we find that it is composed of the nodal linear generators and the two families of lines which cut these generators.

On surfaces of constant mean curvature we have

$$R' + R = K.$$

Using this relation, we can put equation (10) in the form

$$\frac{\partial R}{\partial u} + 3\frac{\partial R}{\partial v}\tan w - 3\frac{\partial R}{\partial u}\tan w - \frac{\partial R}{\partial v}\tan 3w = 0$$

and therefore

$$\frac{\partial R}{\partial u} + \frac{\partial R}{\partial v}\tan 3w = 0$$

differ by ½. Hence we have the theorem due to Ribaucour

On a surface of constant mean curvature, the surfaces osculating lines form an equilateral system.

are readily found on surfaces of the second degree. Consider first the ellipsoid x y and z being measured with respect to the axes of figure, a and b being the parameters of the lines of curvature, the equation is of the form

$$\frac{x^2}{a^2(a-\rho)(a-\sigma)} + \frac{y^2}{b^2(b-\rho)(b-\sigma)} + \frac{z^2}{\rho\sigma} = 1$$

$$y^2 = \frac{b^2(b^2+u)(b^2+v)}{(b^2-c^2)(b^2-a^2)}$$

$$z^2 = \frac{c^2(c^2+u)(c^2+v)}{(c^2-a^2)(c^2-b^2)}$$

Ferner setzung

$$U = 4(a^2+u)(b^2+u)(c^2+u)$$

$$V = 4(a^2+v)(b^2+v)(c^2+v)$$

nun kann

$$F = \frac{u(u-v)}{U}$$

$$f = \frac{-abc(u-v)}{U\sqrt{uv}}$$

i.e. asymptotic lines
together with the family
uv = constant.
But this involves also
$\bar{pp'}$ = Constant
the centre on the tangent plane
at a point (u, v) is given by
the equation
$$\bar{p'} = \frac{uv}{a^2 b^2 c^2}$$
Hence on an ellipsoid
the real principal radius
line through any point
to the point through
that point, the centre of
the ellipsoid
fixed point

Conversely, the normal section taugent to the polhode through any point is superosculated by a circle.

This law been stated by Darboux. It amounts to saying that the polhode is a superosculating line.

The reasoning holds word for word, for the hyperboloids of one and two sheets, if [illegible] of b^2 and a^2. Hence we can [illegible]

On central surfaces of the second degree, the asymptotic lines are superosculating lines, the third

family being lines along which the total curvature of the surface is constant. * Darboux makes this last statement for all surfaces of the second degree, but it does not hold for the hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Since $\frac{dz}{z} = 0$ and $z = 0$ are a pair of the asymptotic lines. Keeping them as parameter lines, the differential equation of the asymptotic lines can be put in the form

$$\left(\frac{x\,dx}{a^2} + \frac{y\,dy}{b^2}\right)\left(L''dx^2 + 2M''dxdy + N''dy^2\right) = 0$$

that is, the asymptotic lines

* Bulletin des Sciences Mathématiques, 2nd series, vol. 2, 1878.

family

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \text{const} = k^2$$

Calling the derivatives of z with respect to x and y p q r s and t as usual. The total curvature at any point is

$$\frac{RR'}{(1+p^2+q^2)^2}$$

which in this case

$$= \frac{u}{ab}\,\frac{\left(1+\frac{ux'}{a^2}\right)\left(1+\frac{uy'}{b^2}\right)}{\left(1+\frac{ux'}{a^2}+\frac{uy'}{b^2}\right)^2}$$

$$= \frac{ab(1+u+c)}{\text{[illegible]}}$$

[illegible] theorem is not constant along the line.

Returning to the general case of oblique axes, we work to find the locus of the centres of geodesic curvature of all curves

a point

If (x, y) is the centre of

Differentiate the fourth of equations (VI) covering the ochoa and the two following

$$\frac{\sqrt{B}}{R'^2}\frac{\partial R'}{\partial u} = -\left(\frac{1}{R}-\frac{1}{R'}\right)\frac{\partial\sqrt{G}}{\partial u}$$

$$\frac{\sqrt{E}}{R^2}\frac{\partial R}{\partial v} = -\left(\frac{1}{R}-\frac{1}{R'}\right)\frac{\partial\sqrt{E}}{\partial v}$$

and we can put the desired equation in the form

$$\frac{\sqrt{E}}{R'^2}\frac{\partial R'}{\partial v}x^3 - 3\frac{\sqrt{G}}{R'^2}\frac{\partial R'}{\partial u}x^2y + 3\frac{\sqrt{E}}{R'^2}\frac{\partial R}{\partial v}xy^2$$

$$-\frac{\sqrt{G}}{R'^2}\frac{\partial R}{\partial u}y^3 + 3\sqrt{EG}\left(\frac{1}{R}-\frac{1}{R'}\right)xy = 0$$

Hence the condition of geodesic curvature of all curves drawn on a surface vanish

This theorem was called

See Richardson in a paper published in the Comptes Rendus 1875

shown that the cubic has a double point at the origin, the two branches being tangent to the lines of coincidence through the

of this surface are

$$\frac{M}{R^{12}}\frac{\partial R'}{\partial v}x^3 - \frac{3N}{R^{12}}\frac{\partial R'}{\partial u}x^2y + \frac{3L}{R^2}\frac{\partial R}{\partial v}xy^2 - \frac{N}{R^2}\frac{\partial R}{\partial u}y^3 =$$

of this equation depend only on the position of the point on the surface

and not on x and y. These
directions are equally well

$$\frac{1}{R^2}\frac{\partial n}{\partial v}dx^2 - 2\frac{\sqrt{\delta}}{R^2}\frac{\partial n}{\partial u}dx\,dy + 2\frac{\sqrt{E}}{R^2}\frac{\partial n}{\partial v}dx\,dy + \frac{\sqrt{G}}{R^2}\frac{\partial n}{\partial u}dy^2 = 0$$

If now we take the equations
giving reciprocal relations
and use

$$dx = \sqrt{E}\,du$$
$$dy = \sqrt{G}\,dv$$

we get it in the form

Comparing (15) and (17), we
see that the roots of one
the negatives of the reciprocals
of the roots of the other; that
is, the asymptotic directions
of the surface (17) are orthog

onal to the superosculat-

It is obvious geometric

ection of the vectors. A
line through the origin,
perpendicular to the super

meet the cubic (17) at infinity.
But this is precisely the
condition that this line
should determine one of
the asymptotic directions
of the cubic.

Equation (18) has the

for real roots whenever
$f(\theta)$ has real roots, and
conversely.

Fixing our attention a
moment on the sections of
a surface of revolution,
tangent to the established
axes have for those sections

and $d\rho_n$

Then since $d\bar{w} = -\left(\frac{1}{\rho_n} - \frac{1}{\rho_{n'}}\right) \sin w \cos w$

$= 0$

Hence on differentiating
for ρ_n cos $\bar{w}$

The wafer conjugation which is tangent to the parabola is superosculated.

On a surface of revolution we have $\frac{dR'}{}$

So that the cubic (17) will break down into a conic together with the $\overline{X}$ axis. A result in accord with the theorem just proved.

The hessian of the cubic

coincident points at the origin, and, in addition to the three points of inflexion of the cubic. These three points lie on

at once on subtraction the hessian from the cubic

$$xy\left(\frac{\partial J_Q}{\partial u}x + \frac{\partial J_E}{\partial v}y + J_{E\,Q}\right) = 0$$

The factors x and y account for six intersections at the origin, so that the points of inflexion must lie on the line

$$\frac{\partial J_Q}{\partial u}x + \frac{\partial J_E}{\partial v}y + J_{E\,Q} = 0$$

and f_{uv} the radical

respectively, then takes the form

$$\frac{x}{g_0} + \frac{y}{g_1} = 1.$$

that is, the intercepts on the axes are g_0 and g_1; in other words the line passes through the centres of geodesic curvature of the α and β lines.

The sections of a surface by its tangent plane are superosculated.

For the centres of curvature of these sections lie on the cubic

Since they are tangent to the asymptotic lines we have

$$y = r \cos(u) \quad \delta$$

Substituting these values

$$\rho \left| -\frac{1}{1-\bar{R}R'} \frac{\partial R'}{\partial v} + \frac{\partial \Omega}{R'} \frac{\partial R'}{\partial u} + \frac{\partial \bar{R}}{R} \left| \bar{R}' \frac{\partial R}{\partial v} - R' \frac{\partial R}{\partial u} \right. \right.$$

above, satisfy equation (1).

But for any value of ω, there is only one value of ω_1, that is, there is only one osculating circle, or circle of curvature, which osculates one section. But we have already found one section whose centre is on the curve. Hence this section is superosculating and the proposition is proved.

An interesting consequence of this theorem is, that if an asymptotic line is a superosculating line it must be straight

of an asymptotic line, and τ referring to a section tangent to it, we have from equations (3') and (11') since ω is the same for the two lines

$$\frac{1}{\tau} - \frac{d\omega}{ds} = \frac{1}{\tau_0} - \left(\frac{d\omega}{ds}\right)_0 = \frac{1}{\tau_0}$$

$$\rho\left(\frac{1}{\tau} - \frac{d\omega}{ds}\right) = \frac{\rho}{\tau_0}$$

If we consider the section of the tangent plane, we have

$$\frac{1}{\tau} = 0$$

and therefore $\rho = \frac{3}{2}\tau_0$.

point of view is important
line, and therefore it must
a point at infinity.

We have found that
equation (17) breaks
into factors in the case of
surfaces of revolution.
In what other cases may
this happen?

In order that the cubic
break down, ω must for some
value of w, become in de-
terminate. If this is the
case, we must have from

therefore, the direction for which $\frac{\Omega}{\omega}$ becomes indeterminate, must be the direction either of an asymptotic line or of a line of curvature.

If it be the direction of an asymptotic line this line must be straight since the normal section is superosculated. The surface is then ruled.

But on a ruled surface all sections tangent to the generator through any point, become merely the generator itself [illegible]

But the cubic does not break down, since all the coincide and are at infinity

The line orthogonal to the generator is then an asymptotic direction of the cubic, as we should expect, since the osculiment generator is a double tangent.

If it be the direction of one of the lines of curvature, then either x or y must be a factor of (17) and we have either

$$\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dx}{dy} = 0$$

If we take the radius vector
and break up into three rectangular
they would be

x y z

and the curve $S(x)$

$$\frac{x}{\rho} \quad \frac{y}{\rho} \quad \frac{z}{\rho}$$

For the surface normal
section through the
curve we will have

$$\frac{3}{2}(k' k')_{min\,max}.$$

Hence we have

$$\frac{f_0'}{2} \quad \frac{3}{2} \quad \frac{f_0'}{2}$$

The radius of geodesic
curvature of a curve, along
which the normal curva-
ture is constant, is
equal to twice the de the
radius of geodesic curva-
ture through it

This theorem is due to
Ribaucour. If the constant
is equal to zero the curve

merely the theorem of Bellavitis which was already used.

On a minima surface one has

$$E = G \qquad R = -R'$$

Putting these relations in equations (16) and (17), they give

$$\frac{\partial E}{\partial u} du^3 + 3\frac{\partial E}{\partial v} du^2 dv + 3\frac{\partial E}{\partial u} du dv^2 + \frac{\partial E}{\partial v} dv^3 = 0$$

$$\frac{\partial E}{\partial v} x^3 - 3\frac{\partial E}{\partial u} x^2 y - 3\frac{\partial E}{\partial v} xy^2 + \frac{\partial E}{\partial u} y^3 - 3Exy = 0$$

They depend on the linear element only of the minima surface so deformed always to minimize.

of osculation and for-
ward.

Hence in such a de-
formation both super-
[illegible] the cubic (17), and so
[illegible]

[illegible]

[illegible]

The osculating plane
of a [illegible] cuts out
a superosculated nodal
of the surface.

For, the centre of the
osculating sphere must
lie on the normal to

the surface, that is,

$$K = f(x, y).$$

Substituting these values in equation (5) we are left with the condition

$$f(x, y) - \frac{1}{2}(x^2 + y^2) = 0$$

must hold between x and y of any osculating plane. But this is exactly the condition that the section cut out by the plane be superosculated.

Conversely, a line, all of whose osculating planes cut out superosculated sections, must be a "σ"

Hence, the osculating spheres are tangent to the

point at a "P" line, we have $dp = 0$; that is, the first curvature is constant. This is not the case however. Solving equations (5) we get the centre of the sphere given by

$$x - \rho \cos\bar{\omega} + \tau \frac{d\rho}{ds} \sin\bar{\omega}.$$

Hence for a "P" line

So that only when we let $\omega = 0$, do we have $dP = 0$.

This shows that if a "D" line is plane, it must be a circle; and, conversely, circular geodesics are "D" lines.

If $\omega = 0$, that is, if the line is a geodesic, we have

$$\frac{dP}{ds} = 0$$

geodesic, its curvature must

To get the envelope of the plane through M which cuts out a superficial

Any section is determined by w and ω, but in the case of superficial sections, when ω is known, it may be found by means of the formula. So that the equation of one of these planes is

$$z\cos\omega\sin\omega\left(\frac{\cos^2\omega}{x''} + \frac{\sin^2\omega}{x'}\right)\left(R - R'\right)$$

$$- z\left(\frac{1}{R''}\frac{\partial R'}{\partial w}\cos^3\omega + \frac{1}{R'^2}\frac{\partial R'}{\partial w}\sin^3\omega\right) =$$

Dividing out by $\cos^3\omega$, we get an equation of the fifth degree in $\tan\omega$, the coefficients of which are all

cone of x y and x.

Then y and x.
ordinary values, that is, since
some one in the plane, not
yet four values of u. The
cone contains by the plane
will, therefore, be of the first
class. If we choose x = 0 only
two values of u are found.
Hence the tangent plane to
the surface is a four fold
plane of the cone.

This was proved by Kum
mer in a memoir "Über
die Zentrafläche der Wellenflächen" published
in Berlin
der Königlichen Akademie

of Astronomiques

The desired envelope will be the discriminant with respect to $\tan\omega$, of the equation of the plane, i.e. the part a cone of degree eight and class four.

To get the cone traced out by the radius through M, of the osculating circle through the point

Any point on one of these radii is given by the equations

$$x = \ldots$$

$$y = \ldots$$

$$z = \ldots$$

where ω and ρ are connected by equation (4)

Then eliminating ρ and ω between these two equations,

$$z\left\{\frac{1}{8R^2}\frac{\partial R}{\partial \omega}y^3 - \frac{1}{8R^2}\frac{\partial R}{\partial \nu}x^3\right\} =$$

$$\left(x\ \frac{x_1}{z_1} + y\ \frac{y_1}{z_1}\right)\left(x\ \frac{x_2}{z_2} + y\ \frac{y_2}{z_2}\right)$$

This is in general a curve of degree four and class six.

To get the curve traced out by the centres of curvature of all the superosculating sections through a point.

This curve will lie on the cone just obtained. It will also lie on the ruled surface which is generated by the axes of the quadric

of curvature of the osculating

One of these axes is given by

$$\sin\omega\cos\bar{\omega} - \cos\omega\cos\alpha\sin\bar{\omega}$$

ω and $\bar{\omega}$ being connected by equations (2) and (3).

Eliminating the parameters between these four equations the required surface takes the

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = \left(\frac{z}{c}\right)^{\frac{2}{3}}$$

The equation of the curve will then be found by

. So we should expect from a geometrical standpoint, the trace of the osculating plane on the tangent plane to simply the cubic (17). That is, the locus of the centers of gradient curvature of the different pencils.

The binormals to the osculating sections lie on a cone of the fifth degree, which is readily found. The binormal is given by

$$\frac{x}{\rho \sin\omega \cos\omega} = \frac{-y}{\cos\omega \cos 2\omega} = \frac{z}{\rho \sin\omega}$$

So that the cone is

$$1 - \frac{\partial R}{} y^3 - \frac{3}{} \frac{\partial R}{} y^2 x + \cdots$$

$$\left(R - \frac{1}{R'}\right)\frac{2yz}{x^2+y^2} = \left(\frac{x^2}{R'^2} + \frac{y^2}{R^2}\right).$$

To find the locus of the projection on the xy plane of the curve of intersection of the surface and the cylinder through any point.

If (x,y) is the projection of any point

$x = \rho \cos\varphi \cos\omega$

$y = \rho \cos\varphi \sin\omega$

and (3) and (6) eliminating ρ and φ and we get the desired locus in the form

$$u_8 + u_7 = 0$$

where u_i is a homogeneous function of degree i.

Hence it is a curve of the

point at the origin; that is,
it is a removable one.

We should expect this, since
for one value of w, there is
only one value of u and there-
fore only one corresponding

away with contact of the
[illegible]

In the neighborhood of
the origin, and [illegible]
[illegible] of the surface in
the form

$$z = \varphi_2(xy) + \varphi_3(xy) + \dots$$

[illegible]
we can put the equation of
the tangent in the form

$$\eta = a_1 x' + a_2 x'^2 + \dots$$

where

$$a_1 = \varphi_1(1, m)$$
$$a_2 = \varphi_2(1, m) + m\varphi_2(1, m)\varphi_2'(1, m)$$
$$a_3 = \varphi_3(1, m) + m(\varphi_1, \varphi_3)' + \tfrac{1}{2} m^2 (\varphi_2^2)'$$
$$+ \dots$$

The problem [illegible]
[illegible] with respect

The [illegible] of any circle of curvature at M may be made to be one like the sphere

$$x^2 + y^2 + z^2 - 2Rz = 0$$

If we take the section of this by the plane

$$y = mx + nz$$

[illegible] of x we get

$$x = \frac{1+m^2}{2R}x^2 + \frac{mn(1+n^2)}{2R^2}x^3 + \frac{(1+m^2)(1+n^2)^2 + 4m^2n^2(1+n^2)}{8R^3}x^4 + \cdots$$

If the section of the surface is to have contact of the fourth order with this circle, the coefficients of

x^2 x^3 and x^4 in the two expansions must be identical Hence we have

$$\Theta_2(u,n) + m\Theta_1(u,n)\Theta_3^2(u,n) \qquad \frac{1}{2}mn.\frac{(1+m^2)}{2}$$

$$\Theta_4 + m(\Theta_1\Theta_3)' + m^2(\Theta_2^3)'' - \frac{(1+m^2)(1+m^2)^2}{8K^2} + \frac{m^2n^2(1+m^2)}{2}$$

If between these three equations we eliminate m and K we will get an equation in n of degree twelve.

But having given n the modulus will be completely determined.

Hence at any point of any surface, there are

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as first and
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